Technical point on proofs —
make sure if you read the proof — everything you say is a known quantity — i.e. defined before, or you know it exists.

$\boxed{\text{Ex}}$ $(\cos(x))' = -\sin(x)$

$$\cancel{\lim_{h\to 0}} \frac{\cos(x+h) - \cos(x)}{h}$$

Fix x. Consider $\frac{\cos(x+h) - \cos(x)}{h}$

$= \ldots$

$= (\quad)\left(\frac{\sin h}{h}\right) + (\quad)(\quad) + (\quad).$

Then, since the limit of each quantity in parentheses as $h\to 0$ above exists, by ACT,

$$(\cos(x))' = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} = (\quad)(\quad) + (\quad)(\quad) + (\quad)$$

$$= -\sin(x).$$

Exercise 7.5.1. (a) Let $f(x) = |x|$ and define $F(x) = \int_{-1}^{x} f$. Find $\boxed{\text{a}}$ piecewise algebraic formula for $F(x)$ for all x. Where is F continuous? Where is F differentiable? Where does $F'(x) = f(x)$?

(b) Repeat part (a) for the function

$$f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2 & \text{if } x \geq 0. \end{cases}$$

(a) $F(x) = \int_{-1}^{x} |t|\,dt = \int_{-1}^{0} |t|\,dt + \int_{0}^{x} |t|\,dt$

by property of integrals.

$$= \int_{-1}^{0} -t\,dt + \begin{cases} \int_0^x t\,dt & \text{if } x \geq 0 \\ \int_0^x -t\,dt & \text{if } x < 0 \end{cases}$$

FTC

$$= -\frac{1}{2}t^2 \Big|_{-1}^{0} + \begin{cases} \frac{1}{2}t^2 \Big|_0^x & \text{if } x \geq 0 \\ -\frac{1}{2}t^2 \Big|_0^x & \text{if } x < 0 \end{cases}$$

$$= \frac{1}{2} + \begin{cases} \frac{1}{2}x^2 & \text{if } x \geq 0 \\ -\frac{x^2}{2} & \text{if } x < 0 \end{cases}$$

$$\boxed{F(x) = \begin{cases} \frac{1}{2} + \frac{1}{2}x^2 & \text{if } x \geq 0 \\ \frac{1}{2} - \frac{1}{2}x^2 & \text{if } x < 0 \end{cases}}$$

(2FTC) $\Rightarrow$ F is continuous ✓

Note also $f(x) = |x|$ is continuous

2FTC $\Longrightarrow$ $F(x) = \int_{-1}^{x} f$ is differentiable everywhere,

and $F'(x) = f(x)$.

<u>To verify</u>: We would have to do 3 calculations

① $x > 0$ $F'(x) = \left(\frac{1}{2} + \frac{1}{2}x^2\right)' \underset{ADT}{=} x$

② $x < 0$ $F'(x) = \left(\frac{1}{2} - \frac{1}{2}x^2\right)' \underset{ADT}{=} -x$.

③ $x = 0$.

$$\frac{F(h) - F(0)}{h}$$

$$= \begin{cases} \dfrac{\frac{1}{2} + \frac{1}{2}h^2 - \frac{1}{2}}{h} & \text{for } h > 0 \\ \dfrac{\frac{1}{2} - \frac{1}{2}h^2 - \frac{1}{2}}{h} & \text{for } h < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}h & \text{if } h > 0 \\ -\frac{1}{2}h & \text{for } h < 0 \end{cases} = \frac{1}{2}|h|.$$

Thus, since $0 \leq \dfrac{F(h) - F(0)}{h} \leq \dfrac{|h|}{2}$

and $\lim_{h\to 0} \frac{|h|}{2} = 0$, By the squeeze thm, $\lim_{h\to 0} \frac{F(h)-F(0)}{h}$ exists and is 0.

$\therefore$ $F'(0)$ exists & is 0.

(b) $f(x) = \begin{cases} 1 & \text{for } x<0 \\ 2 & \text{for } x \geq 0. \end{cases}$

Let $F(x) = \int_{-1}^{x} f$.

$$F(x) = \int_{-1}^{x} f = \int_{-1}^{0} f + \int_{0}^{x} f$$

$$= \int_{-1}^{0} 1 + \begin{cases} \int_0^x 1 & \text{if } x<0 \\ \int_0^x 2 & \text{if } x \geq 0 \end{cases}$$

$$\overset{\text{FTC}}{=} 1 + \begin{cases} x & \text{if } x<0 \\ 2x & \text{if } x \geq 0 \end{cases}$$

$$F(x) = \begin{cases} 1+x & \text{if } x<0 \\ 1+2x & \text{if } x \geq 0 \end{cases}$$

- F is continuous everywhere by 2FTC.
- F is differentiable at x if x<0 by 2FTC., Because f(x) is continuous on $(-\infty,0)\cup(0,\infty)$
- By the formula above, F is diff. at x if x>0.
- We checked that F is not diff'ble at x=0:

Consider $x_n = \frac{1}{n}$ for $n \in \mathbb{N}$, $y_n = -\frac{1}{n}$ for $n \in \mathbb{N}$. $\Bigg\}$ $\lim x_n = 0$, $\lim y_n = 0$

Then $\dfrac{F(x_n)-F(0)}{x_n - 0} = \dfrac{1+2\left(\frac{1}{n}\right)-1}{\left(\frac{1}{n}\right)} = 2$

and $\dfrac{F(y_n)-F(0)}{y_n-0} = \dfrac{1+\left(\frac{-1}{n}\right)-1}{\left(\frac{-1}{n}\right)} = 1$

So $\lim\limits_{n\to\infty} \dfrac{F(x_n)-F(0)}{x_n-0} = 2 \neq \lim\limits_{n\to\infty} \dfrac{F(y_n)-F(0)}{y_n-0}$

$\therefore \lim\limits_{h\to 0} \dfrac{F(h)-F(0)}{h-0}$ does not exist.

F is not diff'ble at $x=0$.

Exercise 7.5.2. Decide whether each statement is true or false, providing a short justification for each conclusion.

(a) If $g = h'$ for some h on $[a, b]$, then g is continuous on $[a, b]$.

(b) If g is continuous on $[a, b]$, then $g = h'$ for some h on $[a, b]$.

(c) If $H(x) = \int_a^x h$ is differentiable at $c \in [a, b]$, then h is continuous at c.

(a) Y 3 N 1 Neither 1

g must satisfy the IVP, but it doesn't have to be continuous.

$$h(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \le 0 \end{cases}$$

$$h'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) & x > 0 \\ 0 & x \le 0. \end{cases}$$

(b) TRUE! g is continuous.

$$\Rightarrow \text{let } h(x) = \int_{\pi/7}^{x} g$$

Then $h'(x) = g(x)$. by 2FTC.

(c) $H(x) = \int_a^x h$ is diff'ble at $c \in [a,b]$ $\Rightarrow$ h is continuous at c.

T 3 (F 2)

eg let $h(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \frac{\pi}{7} e^{5} & \text{if } x = 0 \end{cases}$

Then $H(x) = \int_a^x h = 0$ for all x.

Exercise 7.5.8 (Natural Logarithm and Euler's Constant). Let

$$L(x) = \int_1^x \frac{1}{t}\, dt,$$

where we consider only $x > 0$.

$\frac{1}{x}$ continuous on $(0, \infty)$ 2FTC 2FTC

(a) What is $L(1)$? 0 Explain why L is differentiable and find $L'(x)$. $= \frac{1}{x}$.

(b) Show that $L(xy) = L(x) + L(y)$. (Think of y as a constant and differentiate $g(x) = L(xy)$.)

$L(xy) = \int_1^{xy} \frac{1}{t}\, dt$ (y const),

By chain rule

$\frac{d}{dx}(L(xy)) = y\, L'(xy)$

$= \frac{1}{xy} \cdot y = \frac{1}{x}$

(c) Show $L(x/y) = L(x) - L(y)$.

(d) Let

$$\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - L(n).$$

Prove that (γ_n) converges. The constant $\gamma = \lim \gamma_n$ is called Euler's constant.

(e) Show how consideration of the sequence $\gamma_{2n} - \gamma_n$ leads to the interesting identity

$$L(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

(b) We computed $L'(x) = \frac{1}{x}$

$$\frac{d}{dx}\left(L(xy)\right) = \frac{1}{x}$$

$$\Rightarrow \left(L(xy) - L(x)\right)' = 0$$

$$\Rightarrow L(xy) - L(x) = C$$

Plug in $x=1 \Rightarrow L(y) - L(1) = C$

$$\Rightarrow L(y) = C$$

$$\Rightarrow L(xy) - L(x) = L(y)$$

$$\Rightarrow \boxed{L(xy) = L(x) + L(y).}$$

ⓒ show $L\left(\frac{x}{y}\right) = L(x) - L(y)$.

Pf. $L\left(\frac{x}{y}\cdot y\right) = L\left(\frac{x}{y}\right) + L(y)$

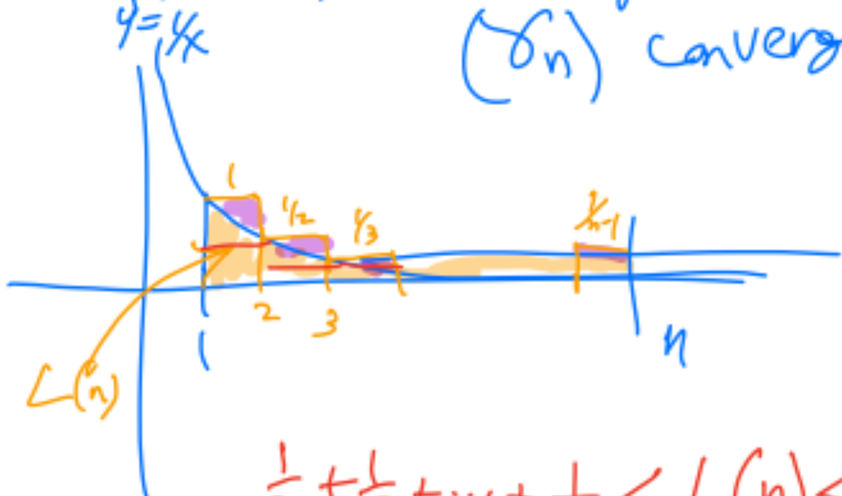
$\Rightarrow \quad L(x) = L\left(\frac{x}{y}\right) + L(y)$

$\Rightarrow \boxed{L\left(\frac{x}{y}\right) = L(x) - L(y)}$.

ⓓ Let $\gamma_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) - L(n)$.

Prove that the sequence (γ_n) converges.

positive
increasing



$\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} < L(n) < 1 + \frac{1}{2} + \dots + \frac{1}{n-1}$

subtract $1 + \frac{1}{2} + \dots + \frac{1}{n}$

$-1 < L(n) - \left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) < -\frac{1}{n}$

$\Rightarrow \quad 1 > \underbrace{\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - L(n)}_{\gamma_n} > \frac{1}{n} \Rightarrow 1 > \gamma_n > 0$

$\therefore (\gamma_n)$ is a bdd increasing seq.

MCT $\Rightarrow$ $\lim \gamma_n = \underline{\gamma}$ exists.
(Euler's Constant)

ⓔ Consider $(\gamma_{2n} - \gamma_n)$,

Find out what's

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots + \frac{1}{2n}$$

is.

$$= \sum_{k=1}^{2n} \frac{1}{k} - 2\sum_{k=1}^{n} \frac{1}{2k}$$

$$\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{2n}\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right)$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$= \sum_{k=1}^{2n} \frac{1}{k} - \sum_{k=1}^{n} \frac{1}{k}$$

$$= \left(\gamma_{2n} + L(2n)\right) - \left(\gamma_n + L(n)\right)$$

$$\gamma_{2n} = \sum^{2n} \frac{1}{k} - L(2n)$$

$$= (\gamma_{2n}) + L(2) + L(n) - \gamma_n - L(n)$$

$$\Rightarrow 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}$$

$$=(\gamma_{2n}-\gamma_n)+L(2)$$

Take $\lim_{n\to\infty}$

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = \underbrace{\lim_{n\to\infty}\gamma_{2n}}_{\gamma} - \underbrace{\lim_{n\to\infty}\gamma_n}_{\gamma} + L(2)$$

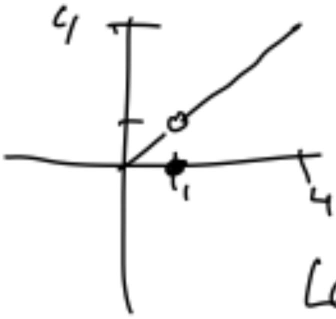
$$= L(2)$$

$$\therefore \boxed{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots = \ln(2)}$$

Questions...

$$f(x) = \begin{cases} 0 & \text{if } x=1 \\ x & \text{otherwise} \end{cases}$$

on $[0,4]$.



Let $P_n = \{\underset{x_0}{0}, \underset{x_1}{\frac{1}{n}}, \frac{2}{n}, \ldots \frac{n-1}{n}, 1, \frac{n+1}{n}, \ldots, \frac{4n}{n}=4\}$

$$x_j = \frac{j}{n} \quad \text{for} \quad 0 \le j \le 4n$$

$$M_k = \sup\{f(x) : x_{k-1} \le x \le x_k\}$$
$$= \sup\{f(x) : \tfrac{k-1}{n} \le x \le \tfrac{k}{n}\}$$
$$= \frac{k}{n}. \qquad f \text{ is increasing.}$$

$$k=n \quad \sup\left(\{x : \tfrac{n-1}{n} \le x < 1\} \cup \{0\}\right) = 1$$

$$k=n+1 \quad \sup\left(\{x : 1 < x \le \tfrac{n+1}{n}\} \cup \{0\}\right) = \frac{n+1}{n}$$

$$m_k = \inf\{f(x) : x_{k-1} \le x \le x_k\}$$
$$= \inf\{f(x) : \tfrac{k-1}{n} \le x \le \tfrac{k}{n}\}$$
$$= \begin{cases} \frac{k-1}{n} & \text{if} \quad k \ne n \text{ or } n+1 \\ 0 & \text{if} \quad \underline{k = n \text{ or } n+1} \end{cases}$$

$$M_k - m_k = \begin{cases} \frac{k}{n} - \frac{(k-1)}{n} = \frac{1}{n} & \text{if } k \ne n \text{ or } n+1 \\ 1 - 0 & \text{if } k = n \\ \frac{n+1}{n} - 0 & \text{if } k = n+1 \end{cases}$$

$$\sum_{k=1}^{4n} (M_k - m_k)\frac{1}{n} = \sum_{k=1}^{n-1} \left(\frac{1}{n}\right)\left(\frac{1}{n}\right) + (1-0)\frac{1}{n} + \left(\frac{n+1}{n}\right)\frac{1}{n}$$

$$+ \sum_{k=n+2}^{4n} \left(\frac{1}{n}\right)\left(\frac{1}{n}\right)$$

$$= \underbrace{\sum_{k=1}^{n-1} \frac{1}{n^2}}_{(n-1)\frac{1}{n^2}} + \frac{1}{n} + \frac{n+1}{n^2} + \underbrace{\sum_{k=n+2}^{4n} \frac{1}{n^2}}_{(4n-(n+2)+1)\cdot\frac{1}{n^2}}$$

$$U(f, P_n) - L(f, P_n) =$$